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The Role of Weak Resonances in AC-Driven Brownian Motion and Rate Processes

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In recent paper a theory of the effect of ac drive on the distribution function and escape rate of a multidimensional underdamped nonlinear oscillator subject to thermal damping and noise was suggested. The approach was based on describing the dynamics in terms of isolated nonlinear resonances and supposing that the noise intensity η is asymptotically small, $\eta \to 0$. In the present work, the case of finite η is considered, when weak resonances cannot be described asymptotically. It is shown that for $p_r/\eta \gg 1$ (p_r is the resonance width) the asymptotic results are valid. For $p_r/\eta \sim 1$, a semiphenomenological theory is developed.

KEY WORDS: Isolated nonlinear resonances; Fokker–Planck equation; thermal averaging; weak-noise asymptotics.

1. INTRODUCTION

In recent work[2] the distribution function of a many-dimensional nonlinear oscillator subject to external ac driving, damping, and noise was established in the limit of large time. Such a distribution function, essentially nonequilibrium in the sense of statistical physics, will be called henceforth a relaxed distribution function (RDF). The presence of damping and external noise models the interaction with the heat bath and corresponds, e.g., to the diffusive approximation in kinetics.[(3)] The main result of the study was the identification of the strong susceptibility of the "tails" of the RDF with the external ac driving in the many-dimensional case. In particular, the function ϕ describing the leading exponential dependence on phase variables of the "tails" of the RDF $\rho = Z\exp(-\phi/T)$ (this represen-

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[2] A brief version of the theory is presented in ref. 1, and a complete version in ref. 2.

tation is asymptotic in the law-temperature limit $T \to 0$) was shown to be strongly perturbed, $\phi_0 - \phi \sim \phi_0$, even by a weak ac driving $\sim \varepsilon$ in the many-dimensional case, while in the one-dimensional case the perturbation is weak, $\phi_0 - \phi \sim \sqrt{\varepsilon}$. The conditions under which this result was obtained were: (1) the exact integrability of unperturbed Hamiltonian oscillator dynamics and (2) asymptotically low temperatures T satisfying $T \ll \sqrt{\varepsilon}$. The amplitude of ac driving ε was also supposed small, $\varepsilon \ll 1$, so that the Hamiltonian dynamics of the driven oscillator can be well described in terms of isolated nonlinear resonances. The importance of the problem can be motivated by possible applications to ac-driven rate processes, presumably, e.g., in laser-stimulated chemical reactions in gases.

The Hamiltonian of an ac-driven integrable oscillator can be conventionally represented[3] in the form

$$H = H_0(\mathbf{J}) + \varepsilon \sum_{l,n} V_{ln}(\mathbf{J}) \cos(\mathbf{l}\boldsymbol{\theta} - n\Omega t) \tag{1}$$

where $\mathbf{J}, \boldsymbol{\theta}$ are action–angle variables for the unperturbed Hamiltonian H_0; Ω is the perturbation frequency; and εV_{ln} is the harmonic amplitude with the integer index $(\mathbf{l}, n)$. Each harmonic generically excites a nonlinear resonance which appears in the resonant action–angle variables I, φ ($\varphi = \mathbf{l}\boldsymbol{\theta} - n\Omega t$ and I is some linear combination of the components of $\mathbf{J}^{(5)}$), as shown in Fig. 1. The amplitude of oscillations of I at the separatrix defines the "resonance width" ΔJ in $\mathbf{J}$ space and is proportional to

[3] In one dimension, a model of the phenomena with exactly the same assumptions as ours was considered in ref. 4.

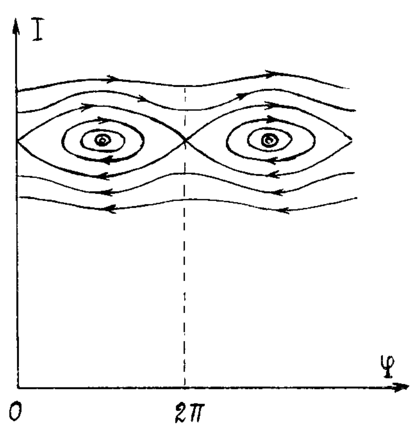

Fig. 1. Phase trajectories in the vicinity of isolated nonlinear resonance in the pendulum approximation (15).

$(\varepsilon\,|V_{ln}|)^{1/2}$. The approach[1,2] was based on the "weak-noise asymptotics" (WNA), which is directly applicable only under the condition $\Delta J^2 \sim \varepsilon\,|V_{ln}| \gg T$. This condition cannot be fulfilled for all resonances present, since generically an infinite number of resonances is excited with arbitrarily small harmonics V_{ln}. However, this does not hamper the evaluation of RDF through our approach[1,2] for fixed damping and asymptotically weak noise (low temperature), since the damping α destroys the resonances with $\varepsilon\, V_{ln}| \lesssim \alpha$, so that in the leading approximation the resonances with $\varepsilon\,|V_{ln}| \ll \alpha$ do not influence the RDF. For finite noise (temperature) and relatively small damping, the approach[1,2] based on the WNA is inapplicable, since there are resonances not destroyed by damping and not satisfying the condition $\varepsilon\,|V_{ln}| \gg T$. In such a situation we need to handle weak resonances not amenable to the WNA description, and this is the object of the present paper.

This paper studies the limits of attempts to increase the range of validity of the theory presented in ref. 1 and 2. In order to make the presentation more self-contained, Section 2 briefly presents the approximations making it possible to reduce the primary FPE (2) to the simpler one (12). For systematic treatment of this reduction see refs. 1 and 2.

Section 3 introduces the "modified" weak-noise asymptotics, differing from weak-noise asymptotic case $\eta \to 0$ (same as $T \to 0$) considered in ref. 1, by allowing the ratio $\alpha(\varepsilon\,|V_{\mathbf{m}}|)^{1/2}/\eta$ to be arbitrary. Thus, except for the last specification, the derivation of Eq. (14) is based on the same assumptions as in refs 1 and 2. New (with respect to refs. 1 and 2) ideas appear only from that point in order to handle Eq. (14) for arbitrary $\alpha(\varepsilon\,|V_{\mathbf{m}}|)^{1/2}/\eta$.

It seems worthwhile to collect all the limitations on the parameters in one place. The first set, which is the same as in refs. 1 and 2, includes the following inequalities:

$$\varepsilon \ll 1; \qquad \alpha \ll \left|\frac{\partial H_0}{\partial \mathbf{x}}\right| \Big/ |\mathbf{p}|; \qquad \eta \ll |\mathbf{p}| \cdot \left|\frac{\partial H_0}{\partial \mathbf{x}}\right|; \qquad \frac{\alpha H_0}{\eta} \gg 1$$

The theory[1,2] was based also on the inequality $\alpha\varepsilon\,|V_{\mathbf{m}}|/\eta \gg 1$, though it was argued qualitatively that the weak-asymptotic results hold under the less restrictive condition $\alpha(\varepsilon\,|V_{\mathbf{m}}|)^{1/2}/\eta \gg 1$. In the present paper this conjecture is put on a firmer basis. For the underdamped case $\varepsilon\,|V_{ln}| \ll \alpha$ a semi-phenomenological approach is developed for determining the influence of resonance on the RDF for arbitrary ratio $(\varepsilon\,|V_{ln}|)^{1/2}/T$.

2. FOKKER–PLANCK EQUATION

The evolution of the distribution density in the phase space is governed by the Fokker–Planck equation (FPE). The primary FPE in coordinate-momentum space is[1,2]

$$\frac{\partial \rho}{\partial t}+\mathbf{p}\frac{\partial \rho}{\partial \mathbf{y}}+\frac{\partial(U_0+\varepsilon\delta U)}{\partial \mathbf{y}}\frac{\partial \rho}{\partial \mathbf{p}}=\frac{\partial}{\partial \mathbf{p}}\left(\alpha\mathbf{p}\rho+\eta\frac{\partial \rho}{\partial \mathbf{p}}\right) \tag{2}$$

where $U_0(\mathbf{y})$ is the unperturbed potential [the unperturbed Hamiltonian is $H_0=\mathbf{p}^2/2+U_0(\mathbf{y})$], $\varepsilon\delta V(\mathbf{y}, t)$ is a periodic perturbation with period $\tau=2\pi/\Omega$, α is the damping decrement, and η is the diffusion coefficient.

Now, we briefly sketch the steps necessary to reduce the FPE (2) to the more tractable form (12) (see refs. 1 and 2 for details). First we introduce the action–angle variables for the unperturbed Hamiltonian $\mathbf{I}, \boldsymbol{\theta}$, so that $H_0=H_0(\mathbf{I})$; suppose the damping α is small relative to the unperturbed frequencies $\nu_x=\partial H_0/\partial I_x$, $\nu_z=\partial H_0/\partial I_z$: $\alpha \ll \nu_x, \nu_z$, and average the FPE over the "fast" phases $\boldsymbol{\theta}$. The "slow" variables are the actions $\mathbf{I}$ and the resonance phase $\varphi=l\theta_x+m\theta_z+n\Omega t$. Limiting ourselves to the two-dimensional case (four-dimensional phase space), we consider, following refs. 1 and 2, only the vicinity of an isolated nonlinear resonance. With the resonance condition $l\nu_x(I_x, I_z)+m\nu_z(I_x, I_z)=n\Omega$, where Ω is the perturbation frequency, the local "thermal-averaged" FPE can be written as

$$\begin{aligned}\frac{\partial \rho}{\partial t}&+\varepsilon V_{\mathbf{m}}\sin\varphi\left(l\frac{\partial \rho}{\partial I_x}+m\frac{\partial \rho}{\partial I_z}\right)+(l\nu_x+m\nu_z-n\Omega)\frac{\partial \rho}{\partial \varphi}\\ &=\frac{\partial}{\partial I_k}\left(-\alpha F_{0k}\rho+\eta G_{0kl}\frac{\partial \rho}{\partial I_l}\right)+(\alpha R_1+\eta R_2)\frac{\partial \rho}{\partial \varphi}\\ &\quad+\eta R_3\frac{\partial^2 \rho}{\partial \varphi^2}+\eta R_{4k}\frac{\partial^2 \rho}{\partial I_k\,\partial \varphi}\end{aligned} \tag{3}$$

Here φ is the Fourier harmonic amplitude of the perturbation $\delta U(\mathbf{I}, \boldsymbol{\theta}, t)$ with the integer index $\mathbf{m}=l, m, n$. The quantities F and G in (3) are the thermal-averaged damping vector and diffusion tensor:

$$\begin{aligned}F_{0k}&=-\frac{1}{(2\pi)^2}\iint_0^{2\pi} d\theta_x\, d\theta_z\, p_i\frac{\partial I_k(\mathbf{y},\mathbf{p})}{\partial p_i}\\ G_{0kl}&=\frac{1}{(2\pi)^2}\iint_0^{2\pi} d\theta_x\, d\theta_z\,\frac{\partial I_k(\mathbf{y},\mathbf{p})}{\partial p_i}\frac{\partial I_l(\mathbf{y},\mathbf{p})}{\partial p_i}\end{aligned} \tag{4}$$

In (4) [as in (3)] the summation by the repeated indices is implied, and the quantities under the integration should be reexpressed in action–angle variables. The quantities R_1 through R_4 in (3) are other averages of the type (4) and are not given explicitly, since to the precision to be considered they can be neglected in (3). To make the calculations less cumbersome, we will consider only the case of separated degrees of freedom in the unperturbed Hamiltonian $U_0(x, z) = U_x(x) + U_z(z)$, where the quantities (4) can be explicitly found to be[1,2]

$$F_{0k} = -I_k$$
$$G_{0kl} = \frac{I_k}{v_k(I_k)} \delta_{kl} \tag{5}$$

The results can be easily generalized for the case of arbitrary (integrable) Hamiltonian H_0.

In the absence of perturbation $\varepsilon = 0$ the FPE (3) has the stationary solution[1,2]

$$\rho = N \exp\left(-\frac{\alpha H_0}{\eta}\right) \tag{6}$$

(N is the normalization factor), which has the familiar Gibbs form with the temperature $T = \eta/\alpha$. What we want to obtain is the steady-state distribution established in the limit of large times in the presence of periodic perturbations. This steady state is the analogue of the stationary distribution in the time-independent system. Such a distribution function, which we will call relaxed (RDF), in the chosen resonance vicinity can only be the stationary solution of the FPE (3). Omitting the terms R_1 through R_4 in (3), which should vanish in the following approximations in the same way as demonstrated in refs. 1 and 2, we obtain the FPE for RDF in the form

$$(lv_x + mv_z - n\Omega)\frac{\partial \rho}{\partial \varphi} + \varepsilon V_m \sin\varphi\left(l\frac{\partial \rho}{\partial I_x} + m\frac{\partial \rho}{\partial I_z}\right)$$
$$= \frac{\partial}{\partial I_x}\left(\alpha I_x + \eta\frac{I_x}{v_x}\frac{\partial \rho}{\partial I_x}\right) + \frac{\partial}{\partial I_z}\left(\alpha I_z + \eta\frac{I_z}{v_z}\frac{\partial \rho}{\partial I_z}\right) \tag{7}$$

The resonance line (resonance center) is defined by

$$\delta\omega = lv_x(I_x) + mv_z(I_z) - n\Omega = 0 \tag{8}$$

In the vicinity of this line let us introduce the resonance action variables I_1, I_2:

$$I_1 = I_e/m$$
$$I_2 = -I_x + \frac{l}{m} I_z \tag{9}$$

The action I_2 is the integral of motion for the Hamiltonian oscillations on the resonance. The FPE in these variables is

$$\delta\omega \frac{\partial \rho}{\partial \varphi} + \varepsilon V_{\mathrm{m}} \sin \varphi \frac{\partial \rho}{\partial I_1}$$
$$= \alpha \left(\frac{\partial}{\partial I_1} I_1 \rho + \frac{\partial}{\partial I_2} I_2 \rho \right)$$
$$+ \eta \left(Q_{11} \frac{\partial^2 \rho}{\partial I_1^2} + 2Q_{21} \frac{\partial^2 \rho}{\partial I_1 \partial I_2} + Q_{22} \frac{\partial^2 \varphi}{\partial I_2^2} \right) + \eta \left(P_1 \frac{\partial \rho}{\partial I_1} + P_2 \frac{\partial \rho}{\partial I_2} \right) \quad (10)$$

where

$$Q_{11} = \frac{I_1}{m v_z}$$
$$Q_{21} = \frac{l I_1}{m v_z} \quad (11)$$
$$Q_{22} = \frac{(-I_2 + l I_1)}{v_x} + \frac{l^2}{m v_z} I_1$$

The quantities P_1, P_2 in the last term in the rhs of (10) are not given explicitly since they will be shown to vanish. Now, since we consider the variation of the RDF in the vicinity of the resonance line $I_1 = I_{10}(I_2)$ defined by the condition (8), it is convenient to introduce one more set of variables $p = I_1 - I_{10}(I_2)$, I_2, φ. The variable p measures the deviations from the resonance center (for $I_2 = \mathrm{const}$), while the variable I_2 for $p = 0$ parametrizes the motion along the resonance line. Introducing the parameter $\kappa(I_2)$, expressed through the tangent of the slope of the resonance line $\operatorname{tg} \gamma = dI_{z0}/dI_x$ as

$$\kappa = \frac{dI_{10}}{dI_2} = \frac{\operatorname{tg} \gamma}{-m + l \operatorname{tg} \gamma}$$

we rewrite the FPE (10) in new variables as

$$\delta\omega \frac{\partial \rho}{\partial \varphi} + \varepsilon V_{\mathrm{m}} \sin \varphi \frac{\partial \rho}{\partial p}$$
$$= \alpha \left[\frac{\partial}{\partial p} (I_1 + p) \rho + \left(\frac{\partial}{\partial I_2} - \kappa \frac{\partial}{\partial p} \right) I_2 \rho \right]$$
$$+ \eta \left\{ Q_{11} \frac{\partial^2 \rho}{\partial p^2} + 2Q_{21} \left(\frac{\partial}{\partial I_2} - \kappa \frac{\partial}{\partial p} \right) \frac{\partial}{\partial p} \rho + Q_{22} \left(\frac{\partial}{\partial I_2} - \kappa \frac{\partial}{\partial p} \right)^2 \rho \right\}$$
$$+ \eta \left[(P_1 - \kappa P_2) \frac{\partial \rho}{\partial p} + P_2 \frac{\partial \rho}{\partial I_2} \right] \quad (12)$$

If the nonlinear resonance is sufficiently narow, i.e., its "width"

$$p_r = \left[2\varepsilon \, |V_{\mathbf{m}}| \Big/ \left| \frac{\partial(\delta\omega)}{\partial p} \right| \right]^{1/2}$$

satisfies the standard condition for the "universal" description[5] $p_r \ll I_{10}, I_{20}$, then the detuning $\delta\omega$ [Eq. (8)] (12) in can be linearized in p around the resonance line $I_1 = I_{10}(I_2)$, so that $\delta\omega \approx \lambda(I_2)\, p$. The quantities κ, Q_{11}, Q_{12}, Q_{22} in the same approximation can be taken on the resonance line, i.e., at $p = 0$, and therefore depend only on I_2.[1,2] The derivative $\partial/\partial I_2$ in (12), and in all the subsequent formulas, implies $\partial/\partial I_2|_{p, \varphi = \text{const}}$.

3. MODIFIED WEAK-NOISE ASYMPTOTICS

In the theory of refs. 1 and 2 the solution of the FPE (12) was found in the WNA $\eta \to 0$. In this asymptotic, the only nontrivial case corresponds to the extremal (most probable trajectory) passing along the resonance line $I_1 = I_{10}(I_2)$. In simpler terms, this is the situation when the particles, in order to undertake a large excursion to the distribution "tail" region, choose the resonance line as their favorite path. Only under this condition does the resonance have a strong effect on RDF. Our goal will be the extension of this approach to the case of a "narrow" resonance, i.e., the study of the limit $\eta \to 0$ for an arbitrarily small value of perturbation and resonance width $\sim \sqrt{\varepsilon}$. Owing to this smallness, the WNA treatment[1,2] is invalid, and a modification is required.

In order to develop the analytical treatment of the problem, we need to use the asymptotic smallness of the parameter $\eta \to 0$. Since we already have one free parameter, the resonance harmonic amplitude $\varepsilon V_{\mathbf{m}}$, which can be arbitrarily small relative to η, the only way of preserving the meaningfulness of the condition $\eta \to 0$ is to suppose the largeness of all the parameters of the problem other than $\varepsilon V_{\mathbf{m}}$, relative to η. So the scale of variation of the harmonic amplitude $V_{\mathbf{m}}(I_2)$, as of all the rest of the parameters of FPE (12), will be supposed to be fixed while $\eta \to 0$. Then, the solution of the FPE (12) in the limit $\eta \to 0$ can be assumed by the same arguments as in refs. 1 and 2, to have the following functional form:

$$\rho(p, I_2, \varphi, \eta) = Z(p, I_2, \varphi, \eta) \exp\left(-\frac{\alpha\phi(I_2)}{\eta} \right) \tag{13}$$

with $\partial Z/\partial I_2 \sim Z$ (i.e., tends to a constant for $\eta \to 0$). The representation (13) corresponds to the WNA applied only in a single direction, along the resonance line (variable I_2). In the transverse direction (the variable p) the variation of RDF is defined by the preexponential factor Z. Substituting similarly to the WNA approach,[1,2] the representation (13) in the FPE (12) and singling out the highest degree of $(1/\eta)$, we arrive at

$$\lambda p \frac{\partial Z}{\partial \varphi} + \varepsilon V_{\mathrm{m}} \sin \varphi \frac{\partial Z}{\partial p} = -\alpha \left[\frac{\alpha(I_2 q - Q_{22} q^2)}{\eta} + (2Q_{21} q - I_{10} - 2Q_{22} \kappa q) \frac{\partial}{\partial p} - (Q_{11} + Q_{22} \kappa^2 - 2Q_{21} \kappa) \frac{\eta}{\alpha} \frac{\partial^2}{\partial p^2} \right] Z \tag{14}$$

where $q = d\phi/dI_2$. Since we will be interested, as in refs. 1 and 2, only in the exponential dependence of RDF on the coordinate along the resonance line, our goal will be the evaluation of $q(I_2)$.

To proceed with the analysis of Eq. (14), consider the underdamped case $\alpha |I_{10}| \ll \varepsilon |V_{\mathrm{m}}|$. The introduction of this limitation is the price we have to pay for allowing the resonance width to be unspecified relative to η. In refs. 1 and 2, when considering a fixed resonance width for asymptotically small η, we were able to handle the arbitrary ratio $|\alpha I_{10}/\varepsilon V_{\mathrm{m}}|$. Quite naturally, widening the range of applicability of the theory in one parameter narrows the range of applicability in another parameter. Still, as we will see later, even this kind of generalization clarifies many aspects of the possibilities of the approach[1,2] to describe the ac-driven distribution functions and escape rates of many-dimensional exactly integrable systems.

In the underdamped limit $\alpha I_{10} \ll \varepsilon |V_{\mathrm{m}}|$, the reduction of Eq. (14) to lower dimensionality through the "thermal averaging" technique (see refs. 1 and 5–7) is straightforward. The function Z has to be supposed constant along the trajectories of the Hamiltonian dynamics ($\alpha = 0$, $\eta = 0$) and Eq. (14) averaged along these trajectories. The Hamiltonian dynamics in our case is described by the pendulum approximation of the resonance Hamiltonian for canonically conjugate variables p, φ,[5]

$$H = \frac{\lambda p^2}{2} + \varepsilon V_{\mathrm{m}} \cos \varphi \tag{15}$$

which corresponds to the Liouville operator

$$\hat{L}_H = \lambda p \frac{\partial}{\partial \varphi} - \varepsilon V_{\mathrm{m}} \sin \varphi \frac{\partial}{\partial p}$$

acting on Z in the lhs of (14). The trajectories of the Hamiltonian (15) are shown in Fig. 1. For brevity, we introduce the quantities

$$\begin{aligned} a &= Q_{22}q^2 - I_2 q \\ b &= 2Q_{22}\kappa q + I_{10} - 2Q_{21}q \\ c &= Q_{11} + Q_{22}\kappa^2 - 2Q_{21}\kappa \end{aligned} \tag{16}$$

and rewrite Eq. (14) as

$$\hat{L}_H Z = \alpha\left(\frac{\alpha a}{\eta} + b\frac{\partial}{\partial p} + c\frac{\eta}{\alpha}\frac{\partial^2}{\partial p^2}\right) Z \tag{17}$$

The "thermal averaging" in (17) can be performed[1,2,6–8] by (1) conducting the differentiations in the rhs of (17) while supposing the function Z to depend on p and φ only in the combination H of (15), and (2) averaging the resulting equation over time, supposing p and φ to depend on time as the Hamiltonian trajectories of (15). The lhs of (17) then becomes zero. Introducing simultaneously the action variable $J(H)$ for the pendulum H in (15)[1,2,6–8] to be the argument of Z instead of H eventually yields

$$\frac{a}{\eta}Z + \frac{d}{dJ}\left[bF + c\frac{\eta}{\alpha}G(J)\frac{d}{dJ}\right]Z = 0 \tag{18}$$

where

$$\begin{aligned} F &= \left\langle \left.\frac{\partial J}{\partial p}\right|_{\varphi = \text{const}} \right\rangle \\ G(J) &= \left\langle \left(\left.\frac{\partial J}{\partial p}\right|_{\varphi = \text{const}}\right)^2 \right\rangle \end{aligned} \tag{19}$$

The symbol $\langle \cdots \rangle$ in (19) implies the averaging over time along the trajectories of the Hamiltonian H. It is important for what follows that the reduction (18) is obtained for arbitrary Hamiltonian H corresponding to the Liouville operator in the lhs of (17), though we are primarily concerned with a particular Hamiltonian (15). In deducing Eq. (18), one would need the general relation

$$\frac{d}{dJ}\left\langle \left(\frac{\partial J}{\partial p}\right)^2 \right\rangle = \left\langle \frac{\partial^2 J}{\partial p^2} \right\rangle \tag{20}$$

and the fact that the quanbtity F does not depend on J, both of which stem from the canonical nature of J and p, φ.[1,2,6,7]

Note that copnsidering the preexponential factor Z and the corresponding variations of RDF in the p direction does not mean that we are trying to construct the RDF, attaining its variations on scales as small as the resonance width $\sim(\varepsilon\,|V_{\mathrm{m}}|)^{1/2}$. Since in the general situation one has to take into account several resonances and due to the formation of small (of the order of the resonance width) stochastic regions in the vicinities of their intersections, this would turn out to be impossible.[1,2] However, to find the parameter q defining the variation of the RDF along the resonance line, we need to consider the next order of approximation, i.e., the equation for Z. The parameter q then will be found from the self-consistency of this equation and "physical" boundary conditions for Z. Such logic is generic in singular perturbation schemes in general and in particular in the thermal averaging[6,7] or so-called fast variable elimination[8] techniques for Fokker–Planck equations.

Equation (18) should be interpreted as a continuity equation. The first term corresponds to a distributed self-consistent source of particles, appearing, as can be easily demonstrated, from the nonzero divergence of the "tangent" flux. Indeed, since this "tangent" flux is the component of the flux j_{I_2} in the continuity form

$$\frac{\partial}{\partial\varphi}\,j_\varphi+\frac{\partial}{\partial p}\,j_p+\frac{\partial}{\partial I_2}\,j_{I_2}=0$$

of the primary FPE (12), the condition $\partial j_{I_2}/\partial I_2=0$ in the leading approximation in powers of $1/\eta$ is the same as $a=0$. The expression under the derivative d/dJ in (18) is therefore the "transverse" flux flowing along the J direction (the net flux across the closed curve $H=\text{const}$ in the p,φ plane). In the WNA solution of Eqs. (12) and (14) (in the limit $\varepsilon V_{\mathrm{m}}=\text{const}$, $\eta\to 0$),[1,2] the "tangent" flux is of a zero divergence, so that the coefficient a in (16) is zero, and q is defined by the condition $a(q)=0$. The zero divergence of the "tangent" flux can be given a graphical explanation. Indeed, for small noise (large resonance width) the probability for the particle to deviate from the resonance center to which it is attracted by damping and reach the separatrix surface $H(I_2,\,p,\,\varphi)=\varepsilon V_{\mathrm{m}}$ [with H in (15)] is exponentially small.[1,2] Therefore, conservation of the number of particles inside the separatrix takes place, leading to the zero divergence of the "tangent" flux. The goal of the present study is the consideration of the "narrow" resonance case, when the particles moving along the resonance line have a substantial probability of "falling out" of resonance.

The action variable J for the pendulum Hamiltonian H of (15) can be easily evaluated in terms of elliptic integrals (see, e.g., ref. 5). The coefficients F, G in (19) can be also expressed through elliptic integrals. In ref. 6

only the one-dimensional case was considered, so that the obtained thermally averaged coincides with our Eq. (18) with the first zero term. In this case, the RDF can be obviously found in quadratures for arbitrary F, G. In our case a is nonzero and the situation is different, since the linear second-order ODE with coefficients that are special functions of the argument cannot be solved explicitly. Therefore, we cannot solve Eq. (18) exactly and some further approximations are needed. As such, we will use the phenomenological change of the exact trajectories of the pendulum Hamiltonian H of (15) shown in Fig. 1 by the "simplified" trajectories shown in Fig. 2. The width in p of region II (inside the separatrix) in Fig. 2 will be taken to be kp_r, where $k \sim 1$ will be a phenomenological constant.

Now, Eq. (17) can be averaged along the "simplified" trajectories, supposing that the Liouville operator $\hat{L}_H$ in the lhs corresponds to these trajectories. The average of the lhs is again zero. The result of averaging the rhs will be rather different in regions I and II. In region I, p is a constant of motion, so that averaging the rhs leaves it unchanged. Thus, in region I of Fig. 2 the function Z satisfies the equation

$$\left(\frac{\alpha a}{\eta} + b\frac{\partial}{\partial p} + c\frac{\eta}{\alpha}\frac{\partial^2}{\partial p^2}\right) Z = 0 \tag{21}$$

Since the coefficients a, b, and c in (21) are independent of p, the solution of Eq. (21) in regions I^a and I^b (above and below the resonance) of Fig. 2 can be written as

$$\begin{aligned} Z_a &= B\exp(-K_1 p_1^{\alpha}/\eta) \\ Z_b &= B\exp(-K_2 p_2^{\alpha}/\eta) \end{aligned} \tag{22}$$

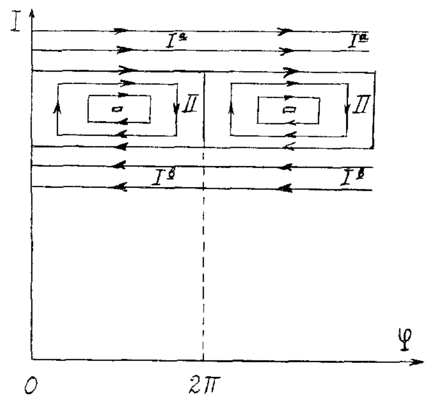

Fig. 2. "Simplified" trajectories approximating the exact ones of Fig. 1.

where the variables $p_1 = p - kp_r$ and $p_2 = p + kp_r$ are introduced for convenience, and the coefficients K_1 and K_2 are

$$K_1 = \frac{b}{2c} + \frac{(b^2+4ac)^{1/2}}{2c}$$
$$K_2 = \frac{b}{2c} - \frac{(b^2-4ac)^{1/2}}{2c} \tag{23}$$

The normalization constant B for Z_a (Z in the region I^a) and Z_b (Z in the region I^b) was taken to be the same, since from the supposed constancy of Z along the trajectories and continuity of Z we have the condition $Z_a(p_1=0)=Z_b(p_2=0)$. The choice of particular solutions (22) from the general solution, which is a linear combination of the two, is defined by the physical sense of the RDF to be found. Indeed, a nonzero positive value of a in (17) and (21) corresponds to a continuously distributed, in the plane p, φ, source of particles. These particles spread in both directions of p, and the unique choice of solutions (22) is the one that provides the oppositely outflowing fluxes in regions I^a and I^b (see also below).

For the averaging of Eq. (17) in region II let us introduce the variable y labeling different trajectories, so that $Z_{\mathrm{II}} = Z_{\mathrm{II}}(y)$. We define this variable to equal the value of p on the upper section of the trajectory rectangle. The quantity y is therefore a uniquely determined function of p and φ in region II. To perform the averaging, we make use of the identity

$$\left\langle \frac{\alpha a}{\eta} + b\frac{\partial}{\partial p} + c\frac{\eta}{\alpha}\frac{\partial^2}{\partial p^2} \right\rangle_{\mathrm{II}}$$
$$= \frac{\alpha a}{\eta} + b\left\langle \frac{\partial y}{\partial p} \right\rangle_{\mathrm{II}} \frac{\partial}{\partial y} + c\frac{\eta}{\alpha}\left[\left\langle \frac{\partial^2 y}{\partial p^2} \right\rangle_{\mathrm{II}} \frac{\partial}{\partial y} + \left\langle \left(\frac{\partial y}{\partial p}\right)^2 \right\rangle_{\mathrm{II}} \frac{\partial^2}{\partial y^2} \right] \tag{24}$$

where the symbols $\langle \cdots \rangle$ stand for the averaging along the trajectories in region II. For the averaging we need to know not only the form of trajectories, as shown in Fig. 2, but also the corresponding time dependence. We will suppose that the velocity of motion in the p, φ plane is time dependent and the same at all four sections of the rectangle. This qualitatively corresponds to the primary pendulum motion (15) and for such motion a piecewise-continuous Hamiltonian can be constructed. From the definition of y we see that $\partial^2 y/\partial p^2 = 0$ for all four sections of the rectangle, $\partial y/\partial p = 0$ at side sections, $\partial y/\partial p = 1$ at upper sections, and $\partial u/\partial p = -1$ at lower ones. From this, for the averages in (24) one obtains

$$\left\langle \frac{\partial^2 y}{\partial p^2} \right\rangle_{\mathrm{II}} = 0$$

$$\left\langle \frac{\partial y}{\partial p} \right\rangle_{\mathrm{II}} = 0 \tag{25}$$

$$\left\langle \left(\frac{\partial y}{\partial p} \right)^2 \right\rangle_{\mathrm{II}} = \frac{1}{2}$$

Thus, the averaged equation in region II is

$$\left(\frac{\alpha a}{\eta} + \frac{c}{2} \frac{\eta}{\alpha} \frac{d^2}{dy^2} \right) Z_{\mathrm{II}}(y) = 0 \tag{26}$$

with the general solution of the form

$$Z_{\mathrm{II}} = A \cos\left[\left(\frac{2a}{c} \right)^{1/2} \frac{\alpha y}{\eta} + \Psi \right] \tag{27}$$

It is not difficult to show that the net flux of particles through the curve $y = \mathrm{const}$ is given for arbitrary $Z(y)$ by the relation

$$j_y = \frac{c}{2} \frac{\eta}{\alpha} \frac{dZ}{dy} \tag{28}$$

Substituting the expression (27) in the formula (28) and requiring the flux j_y at the point $y = 0$ to be zero (since we have no singular sources at this point), we obtain the first constant Ψ of the general solution (27): $\psi = 0$. The second constant A is defined by the continuity condition of Z on the "separatrix" $y = kp_r$, giving

$$Z_a|_{p_1 = 0} = Z_b|_{p_2 = 0} = Z_{\mathrm{II}}|_{y = kp_r}$$

and yielding

$$A \cos\left[\left(\frac{2a}{c} \right)^{1/2} \frac{\alpha}{\eta} kp_r \right] = B \tag{29}$$

The last condition, which will give together with (29) a closed system for the coefficients a, b, and c and subsequently define the magnitude of q, is the continuity of fluxes on the "separatrix" $y = kp_r$. Indeed, the outflowing of the region II flux j_y at $y = kp_r$ should equal the sum of fluxes (the lower one should be taken with the negative sign) in regions I^a and I^b at $p_1 = 0$

and $p_2 = 0$, respectively. The fluxes in region I are given, as follows from the continuity form of Eq. (21), by

$$j_{pI} = \left(b + c \frac{\eta}{\alpha} \frac{\partial}{\partial p} \right) Z \tag{30}$$

From (30), substituting the expressions (22), the required condition is

$$c(K_1 - K_2) B = \frac{c}{2} A \left(\frac{2a}{c} \right)^{1/2} \sin \left[\left(\frac{2a}{c} \right)^{1/2} \frac{\alpha k p_r}{\eta} \right] \tag{31}$$

Exclusing the constants A and B from relations (29) and (31) yields

$$\left[\frac{2}{ac} (b^2 - 4ac) \right]^{1/2} = \tan \left[\left(2 \frac{a}{c} \right)^{1/2} \frac{\alpha k p_r}{\eta} \right] \tag{32}$$

Recall that since the quantities a, b of (16) are functions of q, Eq. (32) defines the quantity q as a function of the resonance width $p_r = (2\varepsilon \, |V_m|/\lambda)^{1/2}$ and the parameters involved. This function cannot be written explicitly due to the transcendental character of (32).

An important result can be seen without the solution of Eq. (32). Indeed, from (32) we see that the factor of the exponential variation of the RDF, $q = d\phi/dI_2$, depends on the resonance width p_r and noise intensity η only as their ratio p_r/η. The same assertion can be easily seen to be valid within the limits of the considered precision also for the primary system without a phenomenological description of the trajectories. This conclusion is of major importance, as explained in the Introduction, for determining the range of applicability of the WNA approach[1,2] to the calculation of the RDF and escape rates of noisy underdamped ac-driven nonlinear oscillator (see also the Conclusions).

Additional implications of Eq. (32) can be obtained from the consideration of the limits $\alpha p_r/\eta \gg 1$ and $\alpha p_r/\eta \ll 1$. Let us begin with the first case. Since the RDF should be everywhere positive, the argument of tan in (32) should lie in the range $[0, \pi/2]$. The coefficient c does not depend on q, and we conclude therefore that $a(q) \to 0$ for $\alpha p_r/\eta \to \infty$. This is evidently the correct result, since in the WNA solution (the limit $p_r = \text{const}$, $\eta \to 0)$[1,2] the magnitude of q is defined by the equation $a(q) = 0$. The two solutions $q = 0$ and $q = I_2/Q_{22}$ of this equation correspond to two possible directions of extremal along the resonance line. We can also obtain the first nonvanishing (in powers of $\alpha p_r/\eta$) correction to the WNA solution. From (32) it is

$$a \approx \frac{\pi^2}{8} \frac{c}{k^2 (\alpha p_r/\eta)^2} \tag{33}$$

Recall once more that k is a phenomenological constant of the order of unity.

To consider the other limiting case, $\alpha p_r/\eta \ll 1$, we observe first that the magnitude of q for arbitrary $\alpha p_r/\eta$ should lie outside of the interval $[0, I_2/Q_{22}]$. Indeed, the physical reason for the influence of the ratio p_r/η on the magnitude of q is the process of the particles "falling out" of the resonance separatrix. This process evidently hampers the particle progress along the resonance line, lowering the probability of reaching the points on this line. The probability (RDF) exponentially decreases in the direction of the extremal along the resonance line,[1,2] with the quantity $|q| = |d\phi/dI_2|$ measuring the rate of the exponential decrease. The sign of q is evidently defined by the extremal direction, and we see that the increase of the modulus of q from the WNA values should shift q out of the range mentioned above. From this, it is clear that the quantity a in (16) is always larger than zero, while c is positive due to the positive definiteness of the diffusion tensor $Q_{\mu\nu}$(11). The quantity $b^2 - 4ac$ entering the lhs of Eq. (32) can be written as

$$b^2 - 4ac = q^2(Q_{21}^2 - 4Q_{22}Q_{11}) + 2q[I_{10}(2Q_{22}\kappa - Q_{21}) + 2I_2(Q_{11} + Q_{22}\kappa^2 - Q_{21}\kappa)] + I_{10}^2 \tag{34}$$

From the expressions (11) for Q_{11}, Q_{12}, and Q_{22} it follows that the coefficient of q^2 in (34) is negative. Since for $a = 0$ the quantity $b^2 - 4ac$ is positive, it is clear that the equation $b^2 - 4ac = 0$ always possesses two real roots $q = q_{01}$, q_{02}. It is these particular roots which evidently satisfy Eq. (32) in the limit $\alpha p_r/\eta \to 0$. We argue, however, that these solutions are "nonphysical"—such values of q are not realized. Indeed, the limit $\alpha p_r/\eta \to 0$ corresponds to the absence of resonance, and the values $q = q_{01}$ and $q = q_{02}$ can be easily shown to follow from the WNA limit of the FPE (7) in the absence of resonance, $\varepsilon V_{\mathrm{m}} = 0$, under the condition of the extremal to pass by the resonance line $I_1 = I_{10}(I_2)$. It is clear, therefore, that these solutions correspond to the fluxes artifically injected at infinity and cannot be realized when constructing the "global" RDF over all the phase space from the minimization of the specific action functionals accounting for the "global" resonance pattern.[1,2] Thus, the limit $\alpha p_r/\eta \to 0$ leads to "nonphysical" solutions which can be discarded, and i n this limit the resonance just disappears—it has no effect on the RDF. It is clear also that, similar to the existence demonstrated in refs. 1 and 2 of a critical damping α_{cr} below which the overdamped resonance in the WNA regime at the point of consideration has no effect on the RDF, for our finite-noise, strongly-underdamped case there exists a critical value of $(\alpha p_r/\eta)_{\mathrm{cr}}$ below which the resonance at the point of consideration is switched off. Note that, just as for α_{cr}, this critical value of $(\alpha p_r/\eta)_{cr}$ will depend on the point on the resonance line and generically also on the global resonance pattern (nonlocality property).[1,2]

Thus, the factor q of the exponential variation of the RDF ρ along the resonance line $q = (\partial\phi/\partial I_2)|_{I_1 = I_{10}(I_2)}$ is defined by the transcendental equation (32) and lies in the range $[q_{10}, 0]$ or $[I_2/Q_{22}, q_{02}]$. The coefficient k in (32) is a phenomenological constant of the order of unity and should be fitted from numerical simulation. For the estimate, the quantity k can be considered unity.

4. CONCLUSIONS

From the results of the present work it follows that the resonance influences the exponential "tails" of the RDF only when its width is larger than or of the order of the temperature η/α, and for $\alpha p_r/\eta \gg 1$ this influence is the same as in the WNA theory[1,2] ($p_r = \text{const}$, $\eta \to 0$). On the other hand, the "staightforward" applicability condition of WNA is $p_r \gg \sqrt{\eta}$. Therefore, in the underdamped case considered in the range $\sqrt{\eta} \gtrsim p_r \gg \eta$, the WNA is directly inapplicable, but gives a correct result for the variation of the RDF along the resonance line. Such an observation is not self-evident and, together with the obtained phenomenological description of the range $\alpha p_r/\eta \sim 1$, is the major result of the present work.

REFERENCES

1. A. L. Gerasimov, *Phys. Lett. A* **135**:92 (1989).
2. A. L. Gerasimov, *Physica D*, in press.
3. E. M. Lifshits and L. P. Pitaevski, *Physical Kinetics* (Nauka, Moscow, 1979), p. 116 [in Russian].
4. S. Zatseping, *Teor. Mat. Fiz.* **1**:55 (1983).
5. B. V. Chirikov, *Phys. Rep.* **52**:263 (1979).
6. J. F. Shönfeld, *Ann. Phys.* **160**:149 (1985).
7. A. L. Gerasimov, Thermal equilibrium of oscillator under the excitation of isolated non-linear resonances, INP Preprint 87-100, Novosibirsk (1987).
8. K. Gardiner, *Handbook of Stochastic Methods* (Springer, 1985).